Parametric Equations and Polar Coordinates

Parameterizing a Circle

A circle of radius r and center (h, k) is parameterized by

$$x = h \pm r\cos\omega t \qquad y = k \pm r\sin\omega t$$

or

$$x = h \pm r\sin\omega t \qquad y = k \pm r\cos\omega t$$

for $\omega \neq 0$.

Parameterizing an Ellipse

An ellipse with center (h, k), width $2a$, and height $2b$ is parameterized by

$$x = h \pm a\cos\omega t \qquad y = k \pm b\sin\omega t$$

or

$$x = h \pm a\sin\omega t \qquad y = k \pm b\cos\omega t$$

for $\omega \neq 0$.

Cycloid

A cycloid is parameterized by the equations

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$$

Projectile Motion

Suppose that an object located a height y_0 above the ground is launched upward with initial speed v_0 at an angle α above the ground. The object follows a parabolic trajectory given by the parametric equations

$$x = v_0(\cos\alpha)t \qquad y = y_0 + v_0(\sin\alpha)t - \tfrac{1}{2}gt^2$$

where g is the acceleration due to gravity, $32\,\mathrm{ft/sec}^2$ or $9.8\,\mathrm{m/sec}^2$.

Derivatives of Parametric Curves

For $\dfrac{\mathrm{d}x}{\mathrm{d}t} \neq 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}x/\mathrm{d}t} \qquad \frac{\mathrm{d}^ny}{\mathrm{d}x^n} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}\right)}{\mathrm{d}x/\mathrm{d}t}$$

Areas with Parametric Curves

For a curve defined parametrically by $x = f(t)$ and $y = g(t)$, where $f(t_a) = a$ and $f(t_b) = b$,

$$\int_a^b y\,\mathrm{d}x = \int_{t_a}^{t_b} g(t)\,f'(t)\,\mathrm{d}t$$

If $g(t_c) = c$ and $g(t_d) = d$, then

$$\int_c^d x\,\mathrm{d}y = \int_{t_c}^{t_d} f(t)\,g'(t)\,\mathrm{d}t$$

Polar Coordinates

A point (x, y) in Cartesian coordinates is represented in polar coordinates by (r, θ), where

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

Symmetry of Polar Graphs

Condition	Symmetry
$f(\theta) = f(-\theta)$	Symmetry about the x-axis
$f(\theta) = f(\pi - \theta)$	Symmetry about the y-axis
$f(\theta) = f(\theta + \pi)$	Symmetry about the origin

Circles with Polar Coordinates

Polar Equation	Cartesian Equation	Graph
$r = k$	$x^2 + y^2 = k^2$	Circle of radius $\lvert k\rvert$ centered at origin
$r = k\cos\theta$	$\left(x - \frac{k}{2}\right)^2 + y^2 = \frac{k^2}{4}$	Circle of radius $\frac{\lvert k\rvert}{2}$ centered at $(x, y) = \left(\frac{k}{2}, 0\right)$
$r = k\sin\theta$	$x^2 + \left(y - \frac{k}{2}\right)^2 = \frac{k^2}{4}$	Circle of radius $\frac{\lvert k\rvert}{2}$ centered at $(x, y) = \left(0, \frac{k}{2}\right)$

Lines with Polar Coordinates

Polar Equation	Cartesian Equation	Graph
$\theta = \tan^{-1} m$	$y = mx$	Line of slope m passing through origin
$r = c\sec\theta$	$x = c$	Vertical line
$r = c\csc\theta$	$y = c$	Horizontal line

Parametric Equations and Polar Coordinates

Limacons

Let a and b be nonzero constants.

Polar Function	Graph
$r = a + b\cos\theta$	Horizontal limacon
$r = a + b\sin\theta$	Vertical limacon

- If $|a| \geqslant |b|$, then the limacon has only one loop.
- If $|a| = |b|$, then the limacon is a cardioid, a heart-shaped curve.
- If $|a| < |b|$, then the limacon has an inner loop.

Roses

Let n be a positive integer.

Polar Function	Graph
$r = \cos n\theta$ or $r = \sin n\theta$ for odd n	Rose with n petals
$r = \cos n\theta$ or $r = \sin n\theta$ for even n	Rose with $2n$ petals

Differentiating Polar Functions

For a differentiable polar function $r = f(\theta)$,

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}, \quad \frac{dx}{d\theta} \neq 0$$

- If r and $\frac{dr}{d\theta}$ have the same sign, then the graph is moving away from the origin; if they have different signs, then the graph is moving toward the origin.
- If x and $\frac{dx}{d\theta}$ have the same sign, then the graph is moving away from the y-axis; if they have different signs, then the graph is moving toward the y-axis.
- If y and $\frac{dy}{d\theta}$ have the same sign, then the graph is moving away from the x-axis; if they have different signs, then the graph is moving toward the x-axis.

Areas with Polar Functions

The area enclosed by the curve $r = f(\theta)$ on $\alpha \leqslant \theta \leqslant \beta$ is

$$A = \tfrac{1}{2}\int_\alpha^\beta [f(\theta)]^2 \, d\theta$$

The area enclosed between $r = f(\theta)$ and $r = g(\theta)$, where $f(\theta) \geqslant g(\theta) \geqslant 0$ on $\alpha \leqslant \theta \leqslant \beta$, is

$$A = \tfrac{1}{2}\int_\alpha^\beta \left([f(\theta)]^2 - [g(\theta)]^2\right) d\theta$$

Arc Length for Parametric Equations

If a curve is traced out once by the differentiable functions $x = f(t)$ and $y = g(t)$ on $\alpha \leqslant t \leqslant \beta$, then

$$L = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Arc Length for Polar Functions

If a polar curve $r = f(\theta)$ is traced out once over $\alpha \leqslant \theta \leqslant \beta$, then

$$L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

Surface Areas of Revolution for Parametric Equations

If a smooth curve parameterized by $x = f(t)$ and $y = g(t)$ does not cross itself on $\alpha \leqslant t \leqslant \beta$, then the surface area of revolution upon rotating the curve about the x-axis is

$$S = \int_\alpha^\beta 2\pi g(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt, \qquad g(t) \geqslant 0$$

The surface area of revolution upon rotating the curve about the y-axis is

$$S = \int_\alpha^\beta 2\pi f(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt, \qquad f(t) \geqslant 0$$

Surface Areas of Revolution for Polar Functions

If a differentiable polar function $r = f(\theta)$ does not cross itself on $\alpha \leqslant \theta \leqslant \beta$, then the surface area of revolution upon rotating the curve about the x-axis is

$$S = \int_\alpha^\beta 2\pi f(\theta)\sin\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta, \quad f(\theta)\sin\theta \geqslant 0$$

The surface area of revolution upon rotating the curve about the y-axis is

$$S = \int_\alpha^\beta 2\pi f(\theta)\cos\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta, \quad f(\theta)\cos\theta \geqslant 0$$